A numerical method for identifying heat transfer coefficient

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Abstract

In this paper, we consider an inverse problem of heat equation with Robin boundary condition for identifying heat transfer coefficient. Combining the method of fundamental solutions with discrepancy principle for the choice of the locations for source points, we give a method for solving the reconstruction problem. Since the resultant matrix is severe ill-conditioning, Tikhonov regularization with L-curve method is employed. Some numerical examples are given for verifying the efficiency and accuracy of the presented method.

1. Introduction

In many engineering contexts, there are many inverse problems for heat equation. The inverse problem for heat diffusion equation can be roughly divided into five principal classes.

1. The first one is backward heat conduction problem. The problem of this kind is also known as reversed-time problem for determination of initial temperature distribution from the known distribution at the final moment of time. The readers can consult the literature, e.g., [1,2].

2. The second one is the problem of identifying the temperature or the flux of temperature at one of the inaccessible boundaries from the over-posed data at the other one which is accessible. This problem is also known as inverse heat conduction problem, e.g., see [3,4].

3. The third one is the problem of reconstructing the coefficient from over-posed data at the boundaries. The readers can consult the literature, e.g., [5,6].

4. The fourth one is the problem of determination of the shape of the unknown boundaries or the crack inside the heat conduction body, e.g., [7,8].

5. The last one is the problem of determining the heat source or sink, e.g., [9,10].

All of these problems are ill-posed problems. Even some of them are highly ill-posed. That is any small change in the input data may result in a dramatic change in the solution. And numerical computation is very difficult. To obtain a stable numerical solution for these kinds of ill-posed problems, some regularization strategies should be applied. Furthermore, some of these problems such as the third one is nonlinear problem, i.e., the solutions of these problems do not depend on the input data linearly. This fact shows that we should solve some nonlinear ill-posed equations. In a word, the difficulty of these kinds of inverse problems lies in the nonlinearity and ill-posedness.

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In the recent years, the method of fundamental solutions (MFS) has been extensively used for solving initial boundary value problems for partial differential equations. The MFS succeeds in solving many problems. However, Schaback writes in [11]: “The method has two drawbacks: it lacks a general error analysis and it needs source points outside the domain which are not easy to place properly.” In order to mollify this criticism we would like to emphasize the advantages of the MFS:

- it is simple and feasible to handle various boundary conditions;
- it is easier to solve inverse problems than boundary element method, finite element method and finite difference method;
- it is applicable to solve more general parabolic-type problems.

In this paper we deal with an inverse problem arising in thermal engineering. Let $X$ be a metallic body with constant conductivity. A portion $C_2$ of its boundary $\partial X$ is inaccessible. In order to detect the unknown heat transfer coefficient on $C_2$, we get information from the Cauchy data (temperature and heat flux) on $C_1$, which is the accessible portion of $\partial X$. The heat equation is valid inside $X$. Here we concentrate on the computational task of constructing approximation to the unknown heat transfer coefficient $c$.

To the authors’ knowledge, most of the current research on inverse coefficient problems for heat equation is devoted to the theoretical works. The main contribution of this paper is to develop a new numerical method for solving inverse boundary coefficient problems.

The outline of this paper is as follows. In Section 2, we present the formulation of the inverse problem and give the MFS to solve the inverse problem, in Section 3 some numerical experiments are done.

2. Mathematical formulation of the problem

A one-dimensional heat conducting body is subject to unknown time dependent heat transfer coefficient at the inaccessible end ($\hat{x} = L$) while the data on temperature and heat flux can be measured at the accessible end ($\hat{x} = 0$). The ambient temperature is assumed to be constant (see Fig. 1).

This gives rise to an inverse boundary value problem, which can be mathematically described by the following governing equation with initial condition and boundary condition:

$$\rho c \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial \hat{x}^2} (\hat{x}, \hat{t}),$$  \hspace{1cm} \text{(2.1)}

$$T(\hat{x}, 0) = T_0(\hat{x}),$$  \hspace{1cm} \text{(2.2)}

$$-\kappa \frac{\partial T}{\partial \hat{x}} (0, \hat{t}) = m(\hat{t}),$$  \hspace{1cm} \text{(2.3)}

$$T(0, \hat{t}) = n(\hat{t}).$$  \hspace{1cm} \text{(2.4)}

The heat transfer coefficient at the boundary $\hat{x} = L$ is to be determined. The definition of $h_1(\hat{t})$ is given by

$$-\kappa \frac{\partial T}{\partial \hat{x}} (L, \hat{t}) = h_1(\hat{t})(T_\infty - T(L, \hat{t})),$$  \hspace{1cm} \text{(2.5)}

where the heat capacity $c$, the density $\rho$, the thermal conductivity $\kappa$, the ambient temperature $T_\infty$ are assumed to be constants.

Define dimensionless transformation

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{\kappa \hat{t}}{\rho c L}, \quad u = \frac{T - T_\infty}{T_\infty}.$$  \hspace{1cm} \text{(2.6)}

Eqs. (2.1)–(2.5) can be rewritten in dimensionless forms, i.e., consider the following third-kind (or Robin) boundary value problem on the rectangle domain $D = \{(x, t) \in (0, 1) \times (0, 1)\}$.

![Fig. 1. One-dimensional inverse boundary coefficient problem.](image-url)
Problem 1:

\[ u_t - u_{xx} = 0, \]  
\[ u(x, 0) = u_0(x), \]  
\[ u_x(0, t) = h(t), \]  
\[ u_x(1, t) = \gamma(t)u(1, t), \]  

where

\[ u_0(x) = \frac{T_0(\tilde{x}) - T_\infty}{T_\infty}, \quad h(t) = -\frac{m(\tilde{t})L}{\kappa T_\infty}, \quad \gamma(t) = \frac{h_1(\tilde{t})L}{\kappa}. \]

Here we try to reconstruct the \( \gamma(t) \) from the extra data

\[ u(0, t) = g(t), \]  
where \( g(t) = \frac{n(t) - T_\infty}{T_\infty} \). This is an ill-posed and nonlinear problem. For the problem (2.7)–(2.11) we seek a solution which is a pair of functions \( \{\gamma(t), u(x, t)\} \) in the spaces \( C[0, 1] \) and \( C^1(D) \cap C^0(\bar{D}) \), respectively. Assume that the smoothness conditions

\[ u_0(x) \in C^1[0, 1], \quad h(t) \in C[0, 1], \quad g(t) \in C[0, 1] \]

and the compatibility conditions

\[ g(0) = u_0(0), \quad h(0) = u'_0(0) \]

hold.

In addition, if \( u(1, t) \neq 0 \) and there exists a function \( \eta(t) \in C^1[0, 1] \) such that

\[
\frac{1}{2\sqrt{\pi}} \int_0^t \frac{\eta(t)}{(t - \tau)^{3/2}} e^{-\frac{\tau}{\sqrt{t - \tau}}} d\tau = g(t) - \frac{1}{\sqrt{\pi t}} \int_0^t u_0(\xi) e^{-\frac{\xi^2}{2t}} d\xi + \frac{1}{\sqrt{\pi}} \int_0^t h(\tau) \sqrt{\tau - \tau} d\tau.
\]

Mathematically, we have the following result on the existence and uniqueness for the inverse problem [12]:

**Theorem 1.** When conditions (2.12)–(2.14) hold, there exists a unique solution of the problem (2.7)–(2.11).

In practical application, the input Cauchy data \( g(t) \) and \( h(t) \) are recorded only at different times \( t_i, i = 1, 2, \ldots, m \), the initial temperature distribution is recorded only at different spatial points \( x_i, i = 1, 2, \ldots, n \). Assume that the measured datum satisfy

\[ \tilde{u}_0(x_i) = u(x_i, 0) + \delta_1 \varepsilon_i^{(1)}, \quad i = 1, 2, \ldots, n; \]  
\[ \tilde{g}(t_i) = u(0, t_i) + \delta_2 \varepsilon_i^{(2)}, \quad i = n + 1, n + 2, \ldots, n + m; \]  
\[ \tilde{h}(t_i) = u_x(0, t_i) + \delta_3 \varepsilon_i^{(3)}, \quad i = n + 1, n + 2, \ldots, n + m, \]

where \( u(x, 0) \) (or \( u_0(x_i) \)), \( u(0, t) \) (or \( g(t_i) \)), \( u_x(0, t) \) (or \( h(t_i) \)) are the exact datum, the \( \varepsilon_i^{(k)}, k = 1, 2, 3 \) are the random errors, \( \delta_k > 0, k = 1, 2, 3 \) are the tolerated noise levels, respectively. We assume that the probability density functions for errors \( (k = 1, 2, 3) \) are

\[ f(\varepsilon_i^{(k)}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \varepsilon_i^{(k)^2}}, \]

i.e., the errors follow the normal probability distribution and the means are \( E(\varepsilon_i^{(k)}) = 0 \), the variances are \( Var(\varepsilon_i^{(k)}) = 1 \). That is to say

\[
\sqrt{\frac{1}{n} \sum_{i=1}^n (u_0(x_i) - E(\tilde{u}_0(x_i)))^2} \leq \delta_1; \]
\[
\sqrt{\frac{1}{m} \sum_{i=n+1}^{n+m} (g(t_i) - E(\tilde{g}(t_i)))^2} \leq \delta_2; \]
\[
\sqrt{\frac{1}{m} \sum_{i=n+1}^{n+m} (h(t_i) - E(\tilde{h}(t_i)))^2} \leq \delta_3.
\]

The fundamental solution [13] of Eq. (2.1) is

\[ K(x, t) = -\frac{1}{2\sqrt{\pi t}} \exp \left( -\frac{x^2}{4t} \right) H(t), \]

where \( H(t) \) is the Heaviside function. Let us make some symbols:
\[ \phi_j(x, t) = K(x - y_j, t - \tau_j), \quad j = 1, 2, \ldots, N; \]  
\[ \phi'_j(x, t) = \frac{\partial K(x - y_j, t - \tau_j)}{\partial x}, \quad j = 1, 2, \ldots, N, \]

where \( \{y_j, \tau_j\}_{j=1}^N \) are the source points located on \( \{x = \text{Res}1 \times (t \in [-\delta t, 1])\}, \{x = \text{Res}2 \times (t \in [-\delta t, 1])\}, \) and \( \{x \in [\text{Res}1, \text{Res}2] \times (t = -\delta t)\}, \) which is the fictitious boundary outside the domain D. It seems that some virtual point heat source constitutes the above fictitious boundary.

Following the idea of the method of fundamental solution and radial basis functions for solving PDE, we assume that the approximate solution for problem (2.7)–(2.9) and (2.11) can be expressed by

\[ u^r(x, t) = \sum_{j=1}^N \beta_j \phi_j(x, t), \]

where \( \beta_j \) are the unknown coefficients to be determined. By collocating Eq. (2.25) into Eqs. (2.15)–(2.17), we obtain the following system of linear algebraic equations:

\[ \sum_{j=1}^N \beta_j \phi_j(x_i, 0) = \bar{u}_0(x_i), \quad i = 1, 2, \ldots, n, \]

and

\[ \sum_{j=1}^N \beta_j \phi_j(0, t_i) = \bar{g}(t_i), \quad i = n + 1, n + 2, \ldots, n + m, \]

\[ \sum_{j=1}^N \beta_j \phi'_j(0, t_i) = \bar{h}(t_i), \quad i = n + 1, n + 2, \ldots, n + m. \]

The values of the unknown coefficients \( \beta_j \) can be obtained by solving the following matrix equation:

\[ A \lambda = \bar{b}. \]

where

\[ A = \begin{pmatrix} \phi_1(x_i, 0) \\ \vdots \\ \phi_N(x_i, 0) \\ \phi'_1(0, t_i) \\ \vdots \\ \phi'_N(0, t_i) \end{pmatrix}_{(n+2m) \times N} \]

and \( i = 1, 2, \ldots, N; j = 1, 2, \ldots, n + m; k = n + 1, n + 2, \ldots, n + m, \)

\[ \lambda = (\lambda_1, \ldots, \lambda_N)^T, \]

\[ \bar{b} = (\bar{u}_0(x_1), \ldots, \bar{u}_0(x_n), \bar{g}(t_{n+1}), \ldots, \bar{g}(t_{n+m}), \bar{h}(t_{n+1}), \ldots, \bar{h}(t_{n+m}))^T. \]

Since the original problem (2.7)–(2.11) is highly ill-posed, the ill-conditioning of matrix A in Eq. (2.29) still persists. In other words, most standard numerical methods cannot achieve good accuracy in solving the matrix equation (2.29) due to the severe ill-conditioning of matrix A. In fact, the condition number of matrix A increases dramatically with respect to the total number of source points \( N. \) Many regularization methods have been developed for solving these kinds of ill-conditioning problems. In our numerical experiments, we use the Tikhonov regularization to solve the matrix equation (2.29). The Tikhonov regularized solution for Eq. (2.29) is defined to be the solution to the following least-squares problem:

\[ \min_{\lambda} \| A \lambda - \bar{b} \|^2 + \alpha^2 \| \lambda \|^2, \]

where \( \| \cdot \| \) denotes the usual Euclidean norm and \( \alpha \) is called the regularization parameter.

The identification of a suitable value of the regularization parameter is crucial and is still under intensive research. A recent result can be found in [14], where the regularization parameter of Tikhonov method is selected by a fixed-point iteration method. In our computation we use the L-curve method, which is a kind of noise-free rules, to determine a suitable value for \( \alpha. \) The L-curve method was first applied by Lawson and Hanson [15], and more recently by Hansen and O’Leary [16], to investigate the properties of regularized systems under different values of the regularization parameter.

In our computation, we used the Matlab code developed by Hansen [17] for solving the discrete ill-conditioned system (2.29). Denote the regularized solution of (2.29) by \( \lambda^\alpha. \) The approximated solution \( u^\alpha \) for problem (2.7)–(2.11) is then given by

\[ u^\alpha(x, t) = \sum_{j=1}^N \beta_j^\alpha \phi_j(x, t). \]

Let

\[ u^\alpha_t(1, t) = g^\alpha(t)u^\alpha(1, t). \]
We can then solve Eq. (2.34) for the $\gamma'(t)$ which approaches to the unknown $\gamma(t)$. This gives the approximation $\gamma''(t)$ for the unknown heat transfer coefficient $\gamma(t)$ of the inverse problem (2.7)–(2.11).

3. Numerical results and comparison with space marching difference method

In this section, we give comparison between MFS and space marching difference method (SMDM) for solving the inverse problem (2.7)–(2.11). The numerical implementation of MFS has been introduced in Section 2. Now we focus on the SMDM for solving (2.7)–(2.11). If $u_0(x) \in C^1[0, 1]$, let $\nu(x, t) = u(x, t) - u_0(x)$, then (2.1)–(2.3), (2.5) become

$$
\nu_t(x, t) = \nu_{xx} + (u_0)'(x), \quad 0 < x < 1, \quad 0 < t < 1,
\nu(x, 0) = 0, \quad 0 < x < 1,
\nu(0, t) = g(t) - u_0(0), \quad 0 < t < 1,
\nu_x(0, t) = h(t) - (u_0)'(0), \quad 0 < t < 1.
$$

![Fig. 2. RMSE profile of the computed $\gamma$ with respect to $\delta t$.](image1)

![Fig. 3. RMSE profile of the computed $\gamma$ with respect to $R$.](image2)
We can solve the above problem by method of lines. Some details can be found in [18,19]. The regularization parameter plays an important role for the ill-posed problems. In SMDM, the time step length is the regularization parameter. After obtaining the values of \( v(x,t) \), we take \( u(x,t) = v(x,t) + u_0(x) \), and let \( \gamma'(t) = \frac{u_0(1,t)}{u(1,t)} \) if we have assumed \( u(1,t) \neq 0 \).

To test the accuracy of the approximate solution, we use the root mean square error (RMSE) and the relative root mean square error (RSE), which are defined as follows:

\[
\text{RMSE}(\gamma) = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} (\gamma(t_i) - \gamma'(t_i))^2};
\]

\[
\text{RSE}(\gamma) = \frac{\sqrt{\sum_{i=1}^{N_t} (\gamma(t_i) - \gamma'(t_i))^2}}{\sqrt{\sum_{i=1}^{N_t} (\gamma'(t_i))^2}},
\]

where \( N_t \) is the total number of the test points for the recovery of \( \gamma(t) \). See Eq. (2.34). \( \gamma(t) \) and \( \gamma'(t) \) are the exact solution and the approximate solution, respectively. In the following numerical experiments, we fix \( N_t = 41 \), the total number \( N = 180 \) of source points. In the case when the measurement data include some random noises, we use noisy data \( \tilde{b} = b + \delta_{\text{max}} \cdot \text{randn(size}(b)) \) where the randn function generates arrays of random numbers whose elements are normally distributed with mean 0 and variance 1, the magnitude \( \delta_{\text{max}} = \max_{i=1,2,3} \{ \delta_i \} \) indicates the total error level.

**Example 1.** Let the exact solution for the problems (2.7)–(2.11) be

\[
\begin{align*}
    u(x,t) &= (x + 1)^2 + 2t, \\
    \gamma(t) &= \frac{2}{2 + t}.
\end{align*}
\]

Now we solve the inverse problem by the MFS and SMDM.

![Fig. 4. Computed results for Example 1: (a)–(b) FSM and (c)–(d) SMDM.](image-url)
I. MFS. For numerical implementation of MFS, we need to choose the parameter $\delta t$, $\text{Res}_1$ and $\text{Res}_2$ properly. First we investigate the convergence of the numerical algorithm with respect to the value of $\delta t$. Here, we fix $n = m = 30$, $\text{Res}_1 = 4.5$, $\text{Res}_2 = 5.5$, $\delta_{\text{max}} = 0.01$. From these choices of the parameters, we first obtain the matrix (2.29). The solution for the least-squares problem (2.32) is then obtained by using Hansen’s matlab codes through the L-curve for finding a suitable regularization parameter. The approximation $\gamma'(t)$ to the unknown boundary coefficient $\gamma(t)$ is then obtained by using the Matlab built-in function fzero to solve Eq. (2.34). Fig. 2 shows that the RMSE of the numerical solution lies in the range of $2 \times 10^{-3} - 10 \times 10^{-3}$ with respect to an increase in the value of $\delta t$. The location of the source points in MFS plays an important role in the accuracy of the method.

We choose the source points based on $|\text{Res}_1 - 0.5| = |\text{Res}_2 - 0.5| = R$, i.e., $\text{Res}_1 = -R + 0.5$ and $\text{Res}_2 = R + 0.5$. To investigate the effect of this choice of $R$ in our computation, we fix $n = m = 30$, $\delta_{\text{max}} = 0.01$, $\delta t = 2$. Fig. 3 displays the RMSE as a function of the parameter $R$. It can be observed from Fig. 3 that there is a large range of values of $R$ for best convergence.

According to the discrepancy principle, we can select the $R$ such that

$$\|A_{\delta t} x - b\| = \delta_{\text{max}}. \quad (3.5)$$

Fig. 4a displays the approximation effect with $\delta t = 2$, $m = n = 30$, $\delta_{\text{max}} = 0.10$, $R = 1.90$, $\text{RMSE}(\gamma) = 0.050$, $\text{RSE}(\gamma) = 6.0\%$. Fig. 4b shows the approximation effect with $\delta t = 2$, $m = n = 20$, $\delta_{\text{max}} = 0.01$, $R = 5.00$, $\text{RMSE}(\gamma) = 0.016$, $\text{RSE}(\gamma) = 1.9\%$.

II. SMDM. Contrary to the direct problem, the use of smaller time steps usually introduce instabilities in the inverse solutions. Thus, a large computational time step is often used in the inverse heat conduction problems. In compu-

![Figure 5](image-url)
tation, we take time step length $\Delta t = 1/(N_t - 1)$, where $N_t = 41$. Fig. 4c displays the approximation effect with $\delta_{\text{max}} = 0.10$, $\text{RMSE}(\gamma) = 0.59$, $\text{RSE}(\gamma) = 69.8\%$. Fig. 4d shows the approximation effect with $\delta_{\text{max}} = 0.01$, $\text{RMSE}(\gamma) = 0.12$, $\text{RSE}(\gamma) = 14.6\%$.

**Example 2.** To investigate the applicability of the proposed methods, we consider a more complicated solution:

$$u(x, t) = 2 \exp(-4t)(\cos 2x + \sin 2x) + 3 \left( t^2 + tx^2 + \frac{1}{12} x^4 \right),$$

$$\gamma(t) = \frac{4 \exp(-4t)(\cos 2t - \sin 2t) + 3(t^2 + 2t + \frac{1}{3})}{2 \exp(-4t)(\cos 2t + \sin 2t) + 3(t^2 + t + \frac{1}{12})}. \quad (3.6)$$

I. FSM. Fig. 5a displays the approximation effect with $\delta t = 0.01$, $m = n = 30$, $\delta_{\text{max}} = 0.001$, $R = 1.20$, $\text{RMSE}(\gamma) = 0.40$, $\text{RSE}(\gamma) = 26\%$. Fig. 5b shows the approximation effect with $\delta t = 0.01$, $m = n = 20$, $\delta_{\text{max}} = 0.001$, $R = 5$, $\text{RMSE}(\gamma) = 0.310$, $\text{RSE}(\gamma) = 20\%$. From this example, we can see that the results are less satisfactory than Example 1.

II. SMDM. Fig. 5c displays the approximation effect with $N_t = 41$, $\delta_{\text{max}} = 0.001$, $\text{RMSE}(\gamma) = 2.90$, $\text{RSE}(\gamma) = 180\%$.

**Example 3.** Consider an example with a heat source term $f(t)$:
\[ u_t(x, t) = u_{xx} + f(t), \quad 0 < x < 1, \quad 0 < t < 1, \]
\[ u(0, t) = u(1, t), \quad 0 < x < 1, \]
\[ u(x, 0) = u_0(x), \quad 0 < x < 1, \]
\[ u_x(0, t) = g(t), \quad 0 < t < 1, \]
\[ u_x(1, t) = \gamma(t)u(1, t), \quad 0 < t < 1. \]

If we take \( f(t) = t, \) \( u_0(x) = (x + 1)^2, \) \( g(t) = 1 + 2t + \frac{1}{2}t^2, \) \( h(t) = 2, \) then the exact solution \( \gamma(t) = \frac{8}{\pi^2 e^{2t}}. \)

I. FSM. Fig. 6a shows the approximation effect with \( \delta t = 2, m = n = 30, \) \( \delta_{\text{max}} = 0.1, R = 2, \) \( \text{RMSE}(\gamma) = 0.05, \) \( \text{RSE}(\gamma) = 5\%. \) Fig. 6b shows the approximation effect with \( \delta t = 2, m = n = 30, \) \( \delta_{\text{max}} = 0.01, R = 2, \) \( \text{RMSE}(\gamma) = 0.02, \) \( \text{RSE}(\gamma) = 2\%. \)

II. SMDM. Fig. 6c displays the approximation effect with \( N_t = 41, \) \( \delta_{\text{max}} = 0.01, \) \( \text{RMSE}(\gamma) = 0.10, \) \( \text{RSE}(\gamma) = 10\%. \)

Since the SMDM is a finite difference method, it is very hard for one to obtain high accuracy at the end points \( t = 0 \) and \( t = 1. \) This fact can be seen from the above numerical experiment. Obviously, the MFS has better computational results than the SMDM for solving this inverse problem.

4. Concluding remarks

Although in this paper, we investigated the one-dimensional case, but the MFS method can easily be applied to the two-dimensional case. At the same time, the MFS method can be extended to the nonlinear case, i.e., the \( \gamma(t) \) in Eq. (2.10) is replaced by \( \gamma(u(1, t)). \) In addition, we can foresee that the results for SMDM will be better if high frequencies of the measured datum are filtered firstly. However, for better accuracy of MFS, the relation between the parameter \( \delta t \) and \( R \) is not visible. The investigation are deferred to future work.

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