Blasius flow of thixotropic fluids: A numerical study

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ABSTRACT

In the present work, laminar, two-dimensional flow of an incompressible thixotropic fluid obeying Harris rheological model is investigated above a fixed semi-infinite plate- the so-called Blasius flow. Assuming that the flow is occurring at high Reynolds number, use will be made of the boundary layer theory to simplify the equations of motion. The equations so obtained are then reduced to a single fourth-order ODE using a suitable similarity variable. It is shown that Harris fluids do not render themselves to a self-similar solution in Blasius flow. A local similarity solution is found which enabled investigating the effects of the model parameters on the velocity profile and wall shear stress at a given location above the plate. Numerical results show that for the Harris model to represent thixotropic fluids, the sign and magnitude of the material parameters appearing in this fluid model cannot be arbitrary.

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1. Introduction

Boundary layer theory is widely regarded as one of the most successful idealizations in the history of fluid mechanics. This is certainly true for Newtonian fluids. That is, for these fluids predictions made using this approximate theory are generally found to be in good agreement with experimental observations [1]. As to non-Newtonian fluids, however, the theory is still regarded as incomplete [2–10]. A major obstacle in extending the theory to such fluid systems is the diversity of their constitutive behavior meaning that each fluid model should be addressed separately. Furthermore, the nonlinearity introduced by their shear-dependent viscosity and/or elasticity often gives rise to a formidable mathematical task which cannot be solved, at times, even numerically. Obviously, the situation becomes much more complicated when the viscosity of the fluid is time-dependent.

Time-dependent fluid systems are quite frequent in industrial applications with the common effect being a drop in viscosity by the progress of time. Complex fluid systems such as drilling muds, foodstuff, paints, cosmetics, pharmaceuticals, suspensions, grease, and the like belong to this class of fluids-the so-called thixotropic fluids. Physiological fluids such as blood, synovial fluid, and mucus may also exhibit thixotropic behavior depending on the time scale of the observation. (One should always bear in mind that, the question of whether thixotropic effects have to be taken into account or not depends essentially on the ratio between the physical time scale involved and the time scale needed by fluid elements to change their viscosity; if this ratio is large, thixotropic effects are expected to have a significant influence.)

A common effect among thixotropic fluids is that their viscosity is decreased even when the shear rate is constant [11,12]. Also, when a thixotropic fluid is subjected to a ramping “up” and “down” of the shear rate, a hysteresis effect is observed. That is, at corresponding shear rates, the shear stress on the “up” curve is greater than the shear stress on the “down” curve. Due to such complexities, working with thixotropic fluids is not an easy task whether it is in the theoretical domain or in the...
experimental domain. This is perhaps why there appears to be virtually no published work addressing boundary layer flows of thixotropic fluids. This should be contrasted with the numerous works addressing the effects of a fluid’s viscoelasticity on the characteristics of its boundary layer [2–10]. As a matter of fact, Harris [13,14] appears to be the only one who has ever tried to address boundary layer flows of thixotropic fluids. In his book, he relied on a simple thixotropic fluid model (the so-called Harris model) to investigate the effects of a fluid’s thixotropic behavior on the characteristics of the momentum boundary layer formed above a fixed plate [14]. Using the technique of similarity solution, he reduced the boundary layer equations into a single ODE. But, the equation so obtained was realized to be too formidable to render itself to an analytical or even numerical solution, and so was left unsolved until now. Therefore, the question of how the characteristics of a boundary layer are affected by a fluid’s thixotropic behavior still remains unanswered. In the present work, we will try to shed some light on this issue.

The ODE derived by Harris in relation to the Blasius flow of thixotropic fluids suffers from a serious error [14]. In the present work we will try to rectify this error and derive the correct form of this ODE, which is turned out to be highly-nonlinear and of fourth-order. We will then try to numerically solve the ODE obtained this way using pseudo-spectral collocation method. From these numerical results, we are going to determine the conditions under which the material parameters appearing in the Harris model can represent thixotropic fluids. It will be shown that for the Harris model to represent true thixotropic fluids, there are some restrictions as to the sign and magnitude of the material constants appearing in this fluid model-such issues have been left completely unaddressed by Harris [13,14].

2. Mathematical formulations

As mentioned above, it is one of the prime objectives of the present work to investigate the effects of a fluid’s thixotropic behavior on the characteristics of its boundary layer. To achieve this goal, it suffices to focus on the momentum boundary layer formed above a flat plate although it should be conceded that there are other options. Laminar flow above fixed flat plates—the so-called Blasius flow—has attracted many researchers in the past perhaps due to its technological importance (e.g. in coating applications). As a matter of fact, since the pioneering work of Blasius on the boundary layer flow of Newtonian fluids above a fixed plate [1] different aspects of the flow have been studied in the past for both Newtonian and non-Newtonian fluids alike. Such studies have addressed the effects of heat and mass transfer into account, at times under the influence of an external magnetic field [15–24]. In these studies, the plate itself has been allowed to be fixed or moving, horizontal or vertical, impervious or porous, and rigid or flexible [15–24]. In the present work, we are going to focus on the momentum transfer aspects of the flow only. We are also going to assume that the plate is rigid, fixed, and semi-infinite (see Fig. 1). Our justification for relying on this simple flow geometry stems from the fact that it is the only geometry for which a boundary layer analysis has been carried out for thixotropic fluids in the past [see Ref. 14]. The preliminary nature of the present work also warrants relying on this simple geometry as a good starting point, with no complications introduced by heat/mass transfer effects.

2.1. Mathematical formulation

To derive the boundary layer equations for any thixotropic fluid, one should start from Cauchy equations of motion. For an incompressible fluid undergoing a laminar two-dimensional flow, under isothermal conditions, the governing equations are:

\[ \rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \]  
(1a)

\[ \rho \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} \]  
(1b)

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \]  
(1c)

where it has tacitly been assumed that there is no pressure gradient involved anywhere in the flow. As to the constitutive equation of the thixotropic fluid of interest [11–14], one has the option to choose them from a long list encompassing models
obtained from: (i) continuum mechanics approach, (ii) microstructural approach, and (iii) structural kinetics approach. The simplest thixotropic fluid models available in the literature—also referred to as phenomenological models—belong to the continuum mechanics approach [11,12]. One can notably mention the thixotropic fluid model proposed by Harris in his paper [13]. Unfortunately, the model was realized to be too cumbersome for use in nonviscometric flows. This is perhaps why Harris himself tried his general thixotropic fluid model only in a suddenly-imposed steady shearing, and also in oscillatory shear which are both regarded as viscometric flows. To deal with unbound flow above a fixed plate (which is a nonviscometric flow) Harris [14] relied on a simpler version of his more versatile thixotropic fluid model, the so-called Simplified Harris (SH) model. In the present work, we are going to rely on the SH model to investigate thixotropic effects on the characteristics of the momentum boundary layer formed above a fixed plate. In this fluid model, the viscosity is allowed to be time-dependent through allowing the second invariant of the deformation-rate tensor to be time-dependent; that is:

\[ \tau_{ij} = 2\mu(\varepsilon_{2d}(t))\varepsilon_{ij}, \]  

(2)

where \( \varepsilon_{2d} \) is the second invariant of the deformation-rate tensor, \( 2d_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i) \). In the SH model, a quadratic form is used for the \( \varepsilon_{2d} \) so that we have,

\[ \varepsilon_{2d} = (2d_{ij})^2 = 4 \left[ \left( \frac{\partial u_i}{\partial x_j} \right)^2 + \frac{1}{2} \left( \frac{\partial u_i}{\partial y} + \frac{\partial u_j}{\partial x} \right)^2 + \left( \frac{\partial v_j}{\partial y} \right)^2 \right] > 0. \]  

(3)

As to the viscosity function, in the SH model we have [14]:

\[ \mu = \mu_0 - R_1 \varepsilon_{2d} + R_2 \frac{D\varepsilon_{2d}}{Dt}, \]  

(4)

where \( D/Dt(=\partial/\partial t + u/\partial x + v/\partial y) \) is the material derivative, and \( R_1 \) and \( R_2 \) are material properties. Although in the General Harris (GH) model [see Ref. 13] \( R_1 \) and \( R_2 \) are allowed to be time-dependent, in the SH model they are both assumed to be constant [14]. With \( R_1 \) and \( R_2 \) being constant, one can argue that in the SH model it is the fluid property \( R_2 \) which accounts for a fluid’s thixotropic behavior (see Eq. (4)).

Harris himself tried the performance of his simple thixotropic fluid model, as represented by Eqs. (2)–(4), in laminar boundary layer flow formed above a fixed semi-infinite plate. Having inserted Eqs. (2)–(4) into Eqs. 1a, 1b, and 1c, he then simplified the equations so obtained using boundary layer approximation assuming that the flow is steady in an Eulerian sense. In an attempt to find a similarity solution, Harris then introduced a suitable similarity variable and reduced the boundary layer equations into a single ODE. The ODE obtained this way turned out to be too formidable to render itself to an analytical or even numerical solution, and so was left unsolved over the years. Thus no conclusion was made by Harris [14] as to the effects of a fluid’s thixotropic behavior on the characteristics of its boundary layer formed above a fixed plate.

In our opinion, with \( R_1 \) and \( R_2 \) being constant and with \( \partial / \partial t \) being zero, the viscosity function as given by Eq. (4) cannot represent thixotropic fluids in their true sense. That is to say that, with these assumptions the viscosity of fluid elements are allowed to change in a Lagrangian sense only. But this is also true for shear-thinning fluids in any non-homogenous flow. It can be argued that, for a rheological model to truly represent thixotropic fluids the viscosity of fluid elements should vary in time even when \( \varepsilon_{2d} \) is a constant (e.g., in steady shear flow). But, because in steady shear we have \( D\varepsilon_{2d}/Dt = 0 \), the term representing thixotropic effects (i.e., the \( R_2 - \text{term} \)) completely drops from Eq. (3). Thus, for the Harris model to represent thixotropic effects in homogenous flows, \( R_1 \) should definitely be a function of time. As a matter of fact, with \( R_1 \) and \( R_2 \) both being constant, Harris model cannot represent shear-thinning fluids only. And, for this to be true \( R_1 \) and \( R_2 \) should have the correct sign, and possibly the correct magnitude issues left completely unaddressed by Harris [13,14]. It can be argued that for the Harris model to represent shear-thinning fluids in steady shear (say, when \( R_2 \) is equal to zero) \( R_1 \) should definitely be positive (see Eq. (3)). As to the sign of \( R_2 \), one can argue that since in flow above a fixed plate \( D\varepsilon_{2d}/Dt \) is negative, for the viscosity to decrease in a Lagrangian sense, \( R_2 \) should also be positive. This argument is certainly true when \( R_1 \) is zero. For non-zero values of \( R_1 \), however, Harris model can still represent shear-thinning fluids even when \( R_2 \) is negative. As such, in the present work we are going to assign both positive and negative values to \( R_2 \) while the sign adopted for \( R_1 \) is going to be negative throughout.

Having decided on the Harris model as the constitutive equation of the thixotropic fluid of interest, we now assume that the flow is occurring at such a high Reynolds number that use can safely be made of the boundary layer approximation. Using this approximate theory, the viscosity function, as given by Eq. (4), can be simplified to [see Appendix A]:

\[ \mu = \mu_0 - 2R_1 \left( \frac{\partial u}{\partial y} \right)^2 + 4R_2 \left[ u \left( \frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right) + v \left( \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial y^2} \right) \right]. \]  

(5)

It is easy to check that using the boundary layer theory the y-momentum equation is completely dropped from the analysis. On the other hand, the x-momentum equation is reduced to [see Appendix A]:

\[ \rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu_0 \left( \frac{\partial^2 u}{\partial y^2} \right) - 6R_1 \left( \frac{\partial u}{\partial y} \right)^2 \left( \frac{\partial^2 u}{\partial y^2} \right) \]

\[ + 4R_2 \left[ \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial^2 u}{\partial x \partial y} \right) + v \left( \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial y^2} \right) + \left( \frac{\partial u}{\partial y} \right)^2 \left( \frac{\partial^2 u}{\partial x \partial y^2} + \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x} \frac{\partial^2 u}{\partial y^2} \right) \right]. \]  

(6)
This is the boundary layer equation for the Harris fluid in any two-dimensional flow with no pressure gradient involved. In the present work, this equation together with the continuity equation (Eq. (1c)), will be used to find the velocity profile for Blasius flow of a Harris fluid. As to the boundary conditions required to solve this equation, we are going to rely on the no-slip and no-penetration conditions at the wall. As an extra boundary condition, the velocity profiles are required to merge smoothly with the free-stream velocity outside the boundary layer. If the need arises, we can further assume that far from the plate (i.e., outside the boundary layer) the fluid is stress-free. Therefore, we have the following four boundary conditions at our disposal:

\[ f(0) = 0; \quad f'(0) = 0; \quad f'(\infty) = 1; \quad f''(\infty) = 0. \]  

(7)

Since the flow is incompressible and two-dimensional, the stream function \( \psi(x,y) \) can be invoked such that we have: \( u = \partial \psi/\partial y, v = -\partial \psi/\partial x \). Now, in a search for a similarity solution, like Harris [14] the following similarity variable is introduced,

\[ \eta(x,y) = y \sqrt{\frac{U_\infty}{\nu_0\alpha}}, \]  

(8)

where \( \nu_0 = \mu_0/\rho \). With this similarity variable, the stream function can be made dimensionless as,

\[ f(\eta) = \frac{\psi}{\sqrt{\nu_0 x U_\infty}}. \]  

(9)

In terms of the dimensionless stream function, \( f \), the boundary layer equation, Eq. (6), becomes (see Appendix B for the details):

\[ K_1(x)f''f'' + K_2(x)f'' \left[ \frac{3}{2} f f''f'' + \frac{1}{2} f''f'' + \frac{1}{2} f''f'' + \frac{1}{2} f''f'' \right] + 2f'' + f'' = 0, \]  

(10)

where \( K_1 \) and \( K_2 \) are dimensionless variables related to \( R_1 \) and \( R_2 \) by:

\[ K_1(x) = -12 \left( \frac{R_1 U_\infty^2}{\rho \alpha x^2} \right); \quad K_2(x) = -8 \left( \frac{R_2 U_\infty^4}{\rho \alpha^2 x^2} \right). \]  

(11)

Since \( K_1 \) and \( K_2 \) are both functions of \( x \), one can conclude that the flow lacks a self-similar solution. Evidently, the lack of a self-similar solution can be attributed to non-Newtonian material constants. Lacking self-similar solution, the best that we can do is to look for a local similarity solution. Such a solution, if it can be found, is still valuable in that it can enable us to investigate separate effects of \( R_1 \) and \( R_2 \) on the velocity profile above the plate. Having found the velocity profile, one can then proceed with calculating the wall shear stress from,

\[ \tau_w = \mu_{|y=0} \left( \frac{\partial u}{\partial y} \right)_{y=0}, \]  

(12)

where, based on Eq. (5) we have,

\[ \mu_{|y=0} = \mu_0 - 2R_1 (\partial u/\partial y)^2. \]  

(13)

In terms of \( f \) the wall shear stress can be written as:

\[ \tau_w = \mu_0 f''(0) - \left( 2R_1 U_\infty^3/\nu_0 \alpha \right) \left[ f''(0) \right]^3 = \mu_0 \left\{ f''(0) + \frac{K_1}{6} \left[ f''(0) \right]^3 \right\}. \]  

(14)

From this equation it can be concluded that at a given \( x \)-location the wall shear stress is affected directly by \( R_1 \) but only indirectly by \( R_2 \) (i.e., through \( f''(0) \) only). As will be shown shortly, \( f''(0) \) is a positive number less than one. Since \( K_1 \) is a negative number, Eq. (14) suggests that for \( \tau_w \) to remain positive all along the plate (as it should) \( K_1 \) cannot be an arbitrary number. That is to say that, for the results to be physically meaningful we should have:

\[ |K_1| > 6/\left[ f''(0) \right]^2. \]  

(15)

Obviously, this dictates a serious limitation on the usefulness of the simplified Harris model for representing true thixotropic fluids.

3. Method of solution

Eq. (10) looks too formidable to render itself to an analytical solution, and so we look for a numerical solution. In order to investigate separate effects of \( R_1 \) and \( R_2 \) on the velocity profile and wall shear stress at a given location along the plate, we are particularly interested in the two asymptotic cases of \( K_1 = 0 \) and \( K_2 = 0 \). Due to the highly-nonlinear nature of Eq. (10), we have found it necessary to linearize it first before relying on a suitable iterative method for its numerical solution. In the
present work, use will be made of the Newton–Kontorovitch method for this purpose [25]. To that end, let’s assume that \( f_i \) and \( f_{i+1} \) are the value of \( f \) at a given location obtained from two successive iterations of \( i \) and \( i+1 \), respectively. Obviously, when convergence is reached \( f_{i+1} - f_i \approx \delta \) is less than a small preset number, say \( 10^{-5} \). By writing Eq. (10) at an iteration level of \( i+1 \), and, then replacing \( f_{i+1} \) by \( \delta + f_i \) the following linear ODE is obtained for the unknown function \( \delta \) and its derivatives at iteration \( i \):

\[
\delta''\left[ \frac{1}{2} K_2 f_i(f_i')^2 \right] + \delta''\left[ \frac{3}{2} K_2 f_i(f_i')^2 + K_2 f_i f_i'' + K_1 (f_i')^2 + 2 \right] + \delta''\left[ 2 K_2 f_i f_i'' + \frac{3}{2} K_2 f_i f_i'' + \frac{1}{2} K_2 f_i (f_i'')^2 + K_2 (f_i')^2 + f_i \right] + \delta''\left[ \frac{3}{2} K_2 f_i f_i'' + \frac{1}{2} K_2 f_i (f_i')^2 + \frac{1}{2} K_2 f_i (f_i')^2 + f_i \right] + \frac{3}{2} K_2 f_i (f_i'^2) + \frac{1}{2} K_2 f_i (f_i')^2 + \frac{1}{2} K_2 (f_i')^2 + f_i' + K_1 K_2 f_i (f_i')^2 = 0.
\]

where we have neglected terms nonlinear in \( \delta \) and its derivatives. From Eq. (16) one can conclude that using Newton–Kontorovitch method the original nonlinear problem (see Eq. (10)) is reduced to solving the following linear ODE for the unknown function \( \delta \):

\[
a_4 \delta'' + a_3 \delta'' + a_2 \delta' + a_1 \delta = A = 0,
\]

where the coefficients \( a_4, a_3, a_2, a_1, a_0, A \) are constant at each iteration level \( i \); they are defined by:

\[
a_4 = \frac{1}{2} K_2 f_i(f_i')^2,
\]

\[
a_3 = \frac{3}{2} K_2 f_i(f_i')^2 + K_2 f_i f_i'' + K_1 (f_i')^2 + 2,
\]

\[
a_2 = 2 K_2 f_i f_i'' + \frac{3}{2} K_2 f_i f_i'' + \frac{1}{2} K_2 f_i (f_i'')^2 + K_2 (f_i')^2 + f_i + 2 K_2 (f_i')^2 + f_i',
\]

\[
a_1 = \frac{3}{2} K_2 f_i (f_i')^2,
\]

\[
a_0 = \frac{1}{2} K_2 f_i (f_i')^2 + \frac{1}{2} K_2 f_i (f_i'')^2 + f_i',
\]

\[
A = \frac{3}{2} K_2 f_i (f_i')^2 + \frac{1}{2} K_2 f_i (f_i'')^2 + \frac{1}{2} K_2 (f_i')^2 + 2 K_2 (f_i')^2 + f_i'' + K_1 (f_i')^2 + 2 f_i'' + f_i'' + K_1 (f_i')^2 + 2 f_i'' + f_i'' + K_1 (f_i')^2.
\]

where the boundary conditions for \( \delta \) are:

\[
\begin{align*}
\delta(0) &= 0 - f_i(0), \\
\delta'(0) &= 0 - f_i'(0), \\
\delta''(\infty) &= 1 - f_i''(\infty), \\
\delta''(\infty) &= 0 - f_i''(\infty).
\end{align*}
\]

Eq. (17) can be solved using the fourth-order Runge–Kutta method and/or the spectral method. We have tried both methods and found consistent results for small values of \( K_1 \) and \( K_2 \). But, for large values of \( K_2 \) it was realized that converged results could be obtained with the spectral method only. As such, in this section this method of solution will be described in some details. In a typical spectral method the function \( \delta(\eta) \) is approximated by a sum of \( N \) base functions \( \tilde{\xi}_n \) so that we have:

\[
\delta(\eta) \approx \delta_n(\eta) = \sum_{n=1}^{N} a_n \tilde{\xi}_n(\eta).
\]

As to the selection of the base (or trial) functions \( \tilde{\xi}_n \) there exists some freedom. In the present work, they will be constructed using Chebyshev polynomials in such a way that they will automatically satisfy the required boundary conditions exactly. Thus we set:

\[
\tilde{\xi}_n(\eta) = T_{n-1}(\eta) - \frac{2(n+1)}{n+2} T_{n-1}(\eta) + \frac{n}{n+2} T_{n+1}(\eta),
\]

where \( T_n \) is the Chebyshev polynomial of degree \( n \) defined by the trigonometric functions \( T_n(\eta) = \cos[n\cos^{-1}(\eta)] \). But, because these polynomials are defined in the interval \([-1, 1]\) whereas in Blasius flow the physical domain is \([0, \eta_{\text{max}}]\) the following transformation is used to recast Eq. (17) into the same interval as the base functions:

\[
\zeta = \frac{2}{\eta_{\text{max}}} \eta - 1.
\]
where $\eta_{\text{max}}$ represents the physical infinity, i.e., the edge of the boundary layer. From our experience with Blasius flow of Newtonian fluids, and also from the preliminary results obtained for SH fluids using different values of $\eta_{\text{max}}$, we have reached to the conclusion that $\eta_{\text{max}} = 10$ can well represent the infinity. With $\eta_{\text{max}} = 10$ Eq. (17) is transformed to:

$$
\frac{a_4}{625} \delta'' + \frac{a_3}{125} \delta'' + \frac{a_2}{25} \delta'' + \frac{a_1}{5} \delta' + a_0 \delta + A = 0,
$$

(23)

where now all differentiations are with respect to $\zeta$. Now, by substituting the approximate solution, $\delta_n(\zeta)$ into Eq. (23) a non-zero residue will be left due to the fact that the proposed solution is not an exact solution. The problem then becomes how to choose the coefficients $a_n$ in such a way that this residue is minimized. In the collocation method used in this work, the residue is forced to become exactly equal to zero at certain points called collocation points [26]. As to the collocation points, use will be made of the so-called Gauss–Lobatto quadrature points $\zeta_j$ defined by $\zeta_j = \cos (j \pi/N)$ where $j = 0, 1, 2, \ldots, N$ [26]. A set of $N + 1$ linear algebraic equations will be obtained this way in the form of $\sum_{n=1}^{N} b_{mn} a_n = 0$. Since these $N$ equations are linear and homogenous, a nontrivial solution exists for the coefficients $a_n$ if, and only if, the determinant of the $N \times N$ matrix $b_{mn}$ vanishes identically. We have realized that grid-independent results can be obtained with $N = 120$. To initiate the iteration process, one has to make a choice on $f''(0), f''(0), f''(0), f''(0)$, and $f''(10)$. In the present work, we have relied on their values known for Blasius flow of Newtonian fluids for this purpose [1].

4. Results and discussions

The code developed in the present work had to be verified first and this was easily done by comparing its output for the case of $K_1 = K_2 = 0$ (i.e., the Newtonian case) with published data [1]. Fig. 2 compares the results obtained in this work, using the collocation method, with those reported in the literature for Blasius flow of Newtonian fluids [1]. As can be seen in this figure, the two sets of data are virtually the same demonstrating that the code developed in this work is working properly. Having validated the code, it was then used to investigate separate effects of $R_1$ and $R_2$ on the velocity profiles and wall shear stress at a given location above the plate.

Fig. 3 shows the results obtained for the velocity profiles as a function of $K_1$ (or, equivalently, as a function of $R_1$) for the case of $K_2 = 0$. This figure also includes the Newtonian case for comparison purposes. Since $R_1$ is a positive number, in Fig. 3 we have allowed $K_1$ to take negative values only. From Fig. 3 it can be concluded that by an increase in $R_1$ velocity is increased at any given location above the plate. This also means that, by an increase in $R_1$ the velocity gradient is increased near the wall. This notion can better be seen in Fig. 4 which shows that $f''(0)$ is increased by an increase in the absolute value of $K_1$. These findings are not surprising realizing the fact that by an increase in $K_1$ the fluid’s viscosity is decreased. What is surprising is the prediction that the wall shear stress is increased by an increase in the absolute value of $K_1$ (see Fig. 5). This rather unexpected prediction has the implication that a decrease in the viscosity of the fluid at the wall (when $K_1$ is increased) is more than compensated by an increase in $f''(0)$ to such an extent that in practice the wall shear stress is increased.

Having studied the effect of $K_1$ on the flow characteristics above the plate, we are now at a stage to address the effect of $K_2$ on the velocity profile and wall shear stress. Fig. 6 shows the results obtained for the velocity profiles as a function of $K_2$ (or, equivalently $R_2$) for the case of $K_1 = 0$. This figure also includes the Newtonian case for comparison purposes. As can be seen in Fig. 6, we have allowed $K_2$ to take both positive and negative values for reasons mentioned in the previous sections. From Fig. 6 it can be concluded that by an increase in $K_2$ the velocity is increased at any given location above the plate. Fig. 7 shows

![Graph](image-url)

**Fig. 2.** A comparison between the results obtained in the present work using the collocation method with those reported in the literature for Blasius flow of Newtonian fluids.
Fig. 3. Effect of $K_1$ on the velocity profiles when $K_2 = 0$.

Fig. 4. Effect of $K_1$ on $f'(0)$ when $K_2 = 0$.

Fig. 5. Effect of $R_1$ on the wall shear stress when $R_2 = 0$.
Fig. 6. Effect of $K_2$ on the velocity profiles when $K_1 = 0$.

Fig. 7. Effect of $K_2$ on $f''(0)$ when $K_1 = 0$.

Fig. 8. Effect of $K_2$ on the wall shear stress when $K_1 = 0$. 
that, while for positive values of $K_2$ the curvature of the velocity profile, i.e., $f''(0)$, is increased in comparison to the Newtonian value, for negative values of $K_2$ it is decreased. Fig. 8 presents the effect of $R_2$ on the wall shear stress. As can be seen in this figure, by an increase in $R_2$ the wall shear stress is increased provided that $R_2$ has a positive sign.

From the results presented in the above figures, one can conclude that an increase in $R_1$ increases the wall shear stress when $R_2 = 0$. On the other hand, an increase in $R_2$ is predicted to decrease the wall shear stress when $R_1 = 0$. Since, in general, $R_1$ and $R_2$ can both be non-zero, therefore it is possible that their combined effect may give rise to an overall drop in the wall shear stress as time progresses. Fig. 9 shows that this scenario can indeed happen in practice for a Harris fluid. As seen in this figure, the $R_2/R_1$ term can overshadow the $R_1/R_2$ term to such an extent that, in practice, the fluid of interest exhibits a wall shear stress which is decaying in time—as expected for a thixotropic fluid.

5. Concluding remarks

In the present work, suitability of Simplified Harris (SH) rheological model was investigated in boundary layer studies of thixotropic fluids. This simple fluid model incorporates two material properties to represent a fluid’s thixotropic behavior. Based on the results obtained in this work, it can be concluded that for this fluid model to truly represent thixotropic effects, at least one of these two material properties should be time-dependent. And when these two material properties are both constant, this rheological model can represent thixotropic fluids only if the second invariant of the deformation-rate tensor is time-dependent. With these two material properties being constant and the second invariant of the deformation-rate tensor being time-independent (in an Eulerian sense) the model can, at best, represent shear-thinning (or shear-thickening) fluids only. And for this to be true, the sign and magnitude of these two material constants should be properly chosen. Because of such restrictions/shortcomings, we cannot recommend this unpopular rheological model for studies related to the flows of thixotropic fluids in practical fluid mechanics problems (e.g., start-up of pipelines transporting thixotropic waxy oils).

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Appendix A. Details of the derivation of Eq. (6)

In this Appendix, the details of how to derive Eq. (6) are presented. For convenience, we have decided to represent $u$ by $u_1$, $v$ by $u_2$, $x$ by $x_1$, and $y$ by $x_2$. We also represent partial derivatives by terms such as $u_{i,j}$, $u_{i,j,j}$ and the like where the first subscript represents the velocity component and the subscripts after the comma means the order of differentiation with respect to each coordinate. For example, we have: $u_{1,1} = \frac{\partial u_1}{\partial x_1}$, $u_{1,22} = \frac{\partial^2 u_1}{\partial x_2^2}$, $u_{2,122} = \frac{\partial^3 u_2}{\partial x_1 \partial x_2^2}$, and the like. In a flow without pressure gradient, the $x$-momentum equation can then be written as (Hint: the $y$-momentum equation is dropped based on the boundary layer approximation):

$$\rho (u_1 u_{1,1} + u_2 u_{1,2}) = -p_1 + \tau_{11},$$

where the repeated $j$ means summation over all its possible values (which is 1 and 2 in a two-dimensional flow). For a Harris fluid, the stress tensor is written as:
\[ \tau_{ij} = 2\muII(t)d_{ij}, \] (A.2)

where for convenience we have replaced the second invariant of the deformation-rate tensor, \( II_{2d} \), by \( II \). Based on the Harris model, for a flow which is steady in an Eulerian sense we have,
\[ \mu(II) = \mu_0 - R_1II + R_2 \frac{D}{Dt}(II) = \mu_0 - R_1II + R_2(u_1II_1 + u_2II_2), \] (A.3)

where,
\[ II = (2d_{ii})^2 = 4(d_{i1}^2 + 2d_{i2}^2 + d_{i3}^2) = 4\left[u_{12}^2 + \frac{1}{2}(u_{i1}^2 + 2u_1u_2u_{11} + u_{21})^2\right]. \] (A.4)

Using boundary layer approximation \( II \) can be simplified to:
\[ II \approx 2u_{12}^2. \] (A.5)

As to the stress terms appearing on the right-hand-side of Eq. (A.1), based on the Harris model we have:
\[ \tau_{ijj} = (2\mu d_{ij})_j = 2\mu d_{ijj} + 2\mu d_{ij}. \] (A.6)

The first term on the RHS of this equation can easily be approximated using boundary layer approximation; that is:
\[ 2\mu d_{ijj} = 2\mu(d_{i1,1} + d_{i1,2}) \approx \mu u_{12}. \] (A.7)

Calculating the last term on the right-hand-side of Eq. (A.6) needs more work to be done. To that end, we rewrite Eq. (A.3) as:
\[ \mu = \mu_0 - R_1(2u_{12}^2) + R_2(u_1II_1 + u_2II_2) \approx \mu_0 - 2R_1u_{12}^2 + R_2(4u_1u_2u_{11} + 4u_2u_1u_{1,22}). \] (A.8)

Therefore, based on the boundary layer theory, the \( x \)-momentum equation becomes:
\[ \rho(u_1u_{11} + u_2u_{12}) = \mu_0u_{1,22} - 6R_1(u_{12}^2)^2 + 4R_2 \left[\frac{u_{12}^2u_{11} + u_1u_2}{(u_{12})^2}(u_{11,1} + u_{12,1} + u_{12,2})\right]. \] (A.9)

This is the boundary layer equation for the flow of the Harris thixotropic fluid model above a fixed plate. Together with the continuity equation they constitute a system of two coupled PDEs which should be solved for the unknown velocity components \( u_1 \) and \( u_2 \).

**Appendix B. Details of the derivation of Eq. (10)**

In this Appendix, the details of how to derive Eq. (10) are presented. As previously mentioned, in an attempt to find a similarity solution we can introduce the following similarity variable,
\[ \eta(x,y) = \sqrt[3]{\frac{U_c}{V_0}} \] (B.1)

and then non-dimensionalize the stream function \( \psi(x,y) \) as:
\[ f(\eta) = \frac{\psi(x,y)}{\sqrt{V_0xU_c}}. \] (B.2)

In terms of \( f \), the velocity terms appearing in Eq. (A.9) can be written as:
\[ u_1 = \psi_2 = Uf', \] (B.3)
\[ u_2 = -\psi_x = \frac{1}{2} V\sqrt{\frac{V_0U}{x}} \eta f' - \frac{1}{2} V\sqrt{\frac{V_0U}{x}} f', \] (B.4)
\[ u_{1,1} = \psi_{1,2} = -\frac{U}{2x} \eta f'', \] (B.5)
\[ u_{2,2} = -\psi_{1,2} = \frac{U}{2x} \eta f'', \] (B.6)
\[ u_{1,2} = \psi_{2,2} = U \sqrt{\frac{U}{V_0x}} f'^{2}, \] (B.7)
\[ u_{1,2} = \psi_{2,2} = U \sqrt{\frac{U}{V_0x}} f'^{2}, \] (B.8)
\[ u_{1,12} = \psi_{1,22} = -\frac{U^2}{V_0x^2} \left(f'' + \frac{1}{2} \eta f''\right), \] (B.9)
\[ u_{1,222} = \psi_{2222} = \frac{U^2}{v_0 x} \sqrt{\frac{U}{v_0 x}}, \]  
(B.10)

\[ u_{1,222} = \psi_{2222} = \frac{U^2}{v_0 x} \sqrt{\frac{U}{v_0 x}}. \]  
(B.11)

Having inserted these equations into Eq. (A.1) we will end up with,

\[ K_1(x)f'''' + K_2(x)f'''' + \frac{3}{2}f'''' + \frac{1}{2}f''' + \frac{1}{2}f'' + f'''' = 0, \]  
(B.12)

where we have,

\[ K_1(x) = -12 \left( \frac{R_s U^3}{\rho x^2} \right); \quad K_2(x) = -8 \left( \frac{R_s U^2}{\rho x^2} \right). \]  
(B.13)

References