SOIL DAMPING FORMULATION IN NONLINEAR TIME
DOMAIN SITE RESPONSE ANALYSIS

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Nonlinear time domain site response analysis is used to capture the soil hysteretic re-
sponse and nonlinearity due to medium and large ground motions. Soil damping is
captured primarily through the hysteretic energy dissipating response. Viscous damp-
ing, using the Rayleigh damping formulation, is often added to represent damping at
very small strains where many soil models are primarily linear. The Rayleigh damping
formulation results in frequency dependent damping, in contrast to experiments that
show that the damping of soil is mostly frequency independent. Artificially high damp-
ing is introduced outside a limited frequency range that filters high frequency ground
motion. The extended Rayleigh damping formulation is introduced to reduce the over-
damping at high frequencies. The formulation reduces the filtering of high frequency
motion content when examining the motion Fourier spectrum. With appropriate choice
of frequency range, both formulations provide a similar response when represented by
the 5% damped elastic response spectrum.

The proposed formulations used in non-linear site response analysis show that the
equivalent linear frequency domain solution commonly used to approximate non-linear
site response underestimates surface ground motion within a period range relevant
to engineering applications. A new guideline is provided for the use of the proposed
formulations in non-linear site response analysis.

Keywords: Wave propagation; viscous damping; deep deposits, time-domain; frequency-
domain, nonlinear, equivalent linear, site response.

1. Introduction

One-dimensional (1D) site response analysis is commonly performed to account for
local site effects on ground motion propagation during an earthquake [Idriss, 1968;
Roesset, 1977; Idriss, 1990; Kramer, 1996; Hashash and Park, 2001; Borja et al.,
2002]. Vertically propagating horizontal shear waves (SH waves) approximate the
ground motion, and horizontal soil layers represent site stratigraphy. Soil behaviour
is approximated as a Kelvin-Voigt solid with a linear elastic shear modulus and
viscous damping. The solution of the wave propagation equation is performed in
either frequency or time domain.

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For strong vibrations (medium and large earthquakes), the linear elastic solution is no longer valid since soil behavior is inelastic, non-linear and strain dependent. Equivalent linear analysis, performed in the frequency domain, has been developed to approximate the nonlinear behavior of soil [Schnabel and Idriss, 1972]. The frequency domain solution of wave propagation provides the exact solution when the soil response is linear. The equivalent linear method approximates nonlinear behaviour by incorporating a shear strain dependent shear modulus and damping soil curves. However, a constant linear shear modulus and damping at a representative level of strain is used throughout the analysis.

In nonlinear analysis, the dynamic equation of motion is integrated in the time domain and the nonlinear soil behaviour can be accurately modelled. However, current formulations of non-linear site response analysis contain an additional viscous damping term that does not always provide an accurate result for simulation of ground motion propagation in deep soil deposits. This limitation is relevant for nonlinear site response analysis for regions with thick soil deposits including the Upper Mississippi Embayment [Hashash and Park, 2002] in the Central United States. The Embayment is a deep, up to 1 km thick, sedimentary basin, which overlies the New Madrid Seismic Zone (NMSZ) [Ng et al., 1989; Shedlock and Johnston, 1994].

2. Numerical Formulation for One-Dimensional Site Response Analysis

The 1D equation of motion for vertically propagating shear waves through an unbounded medium can be written as:

\[ \rho \frac{\partial^2 u}{\partial z^2} = \frac{\partial \tau}{\partial z}, \]  

(1)

where \( \rho \) = density, \( \tau \) = shear stress, \( u \) = displacement and \( z \) = depth below the ground surface.

Soil behavior is approximated as a Kelvin-Voigt solid. The shear stress-shear strain relationship is expressed as:

\[ \tau = G \gamma + \eta \frac{\partial \gamma}{\partial t}, \]  

(2)

where \( G \) = shear modulus, \( \gamma \) = shear strain and \( \eta \) = viscosity.

Substituting Eq. (2) into Eq. (1) results in:

\[ \rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial z^2} + \eta \frac{\partial^3 u}{\partial z^2 \partial t}. \]  

(3)

2.1. Frequency domain solution

Equation (3) can be solved for a harmonic wave propagating through a multi-layered soil column [Schnabel et al., 1972], as shown in Fig. 1(a). Introducing a
The motion at any layer can be easily computed from the motion at any other layer (e.g. input motion imposed at the bottom of the soil column) using the transfer function that relates displacement amplitude at layer $i$ to that at layer $j$:

$$ F_{ij}(\omega) = \frac{|u_i|}{|u_j|} = \frac{A_i(\omega) + B_i(\omega)}{A_j(\omega) + B_j(\omega)}. $$

$|\ddot{u}| = \omega |\dot{u}| = \omega^2 |u|$ for harmonic motions and the transfer function can be used to compute accelerations and velocities.

For a harmonic shear strain of the form:

$$ \gamma = \gamma_0 \sin \omega t, $$

$$ \mathbf{Layer} \quad \begin{array}{c|c|c|c|c} \hline \text{Layer} & G_i \rho_i Z_i & m_i/2 & k_i c_1 & h_i \\ \hline 1 & G_1 \rho_1 Z_1 & (m_1 + m_2)/2 & k_1 c_1 & h_1 \\ \hline 2 & G_2 \rho_2 Z_2 & (m_2 + m_3)/2 & k_2 c_2 & h_2 \\ \hline m & G_m \rho_m Z_m & m = \rho_m h & k_m & h_m \\ \hline m+1 & G_{m+1} \rho_{m+1} Z_{m+1} & m_{m+1}/2 & k_{m+1} c_{m+1} & h_{m+1} \\ \hline n & G_n \rho_n Z_n & m_n/2 & k_n c_n & h_n \\ \hline \end{array} \end{array} $$

Fig. 1. Idealised soil stratigraphy (a) Layered soil column (used for frequency domain solution) and (b) multi-degree-of-freedom lumped parameter idealisation (used for time domain solution).
the energy dissipated in a single cycle is:

\[ E_D = \int_{t_0}^{t_0+2\pi/\omega} \tau d\gamma = \int_{t_0}^{t_0+2\pi/\omega} \frac{\partial\gamma}{\partial t} dt = \pi\eta\omega\gamma_0^2. \] (8)

Equation (8) indicates that the dissipated energy is a function of the frequency of the loading. However, experiments show that the damping of soil material is nearly constant over the frequency range of interest in engineering applications [Kramer, 1996]. It is therefore more realistic to express the viscosity in terms of the frequency independent damping ratio \( \xi \). The damping ratio is defined as:

\[ \xi = \frac{E_D}{4\pi E_s} = \frac{\eta\omega}{2G}, \] (9)

where \( E_s = \frac{1}{2}G\gamma_0^2 \).

Rearranging Eq. (9) to eliminate frequency dependence results in

\[ \eta = \frac{2G}{\omega} \xi. \] (10)

When substituting Eq. (10) into Eq. (3), the wave equation becomes independent of frequency for a harmonic loading with a circular frequency of \( \omega \).

Since the solution for an arbitrary loading is performed by transforming the motion into a finite sum of harmonic motions using the Fourier transform, the damping of the system becomes independent of the frequency of the input motion due to the frequency independent viscosity shown in Eq. (10).

### 2.2. Time domain analysis

In time domain analysis, the unbounded medium shown in Fig. 1(a) is idealized as a discrete lumped mass system, as shown in Fig. 1(b). The wave propagation equation, Eq. (3), is written as:

\[ [M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -[M]\{I\}\ddot{u}_g, \] (11)

where \([M]\) = mass matrix; \([C]\) = viscous damping matrix; \([K]\) = stiffness matrix; \( \{\ddot{u}\} \) = vector of nodal relative acceleration; \( \{\dot{u}\} \) = vector of nodal relative velocities; and \( \{u\} \) = vector of nodal relative displacements. \( \ddot{u}_g \) is the acceleration at the base of the soil column and \( \{I\} \) is the unit vector. Equation (11), is solved numerically at each time step using the Newmark \( \beta \) method [Newmark, 1959]. The Newmark \( \beta \) (average) method is unconditionally stable and does not introduce algorithmic damping [Chopra, 1995].

Each individual layer \( i \) is represented by a corresponding mass, a spring, and a dashpot for viscous damping. Lumping half the mass of each of two consecutive layers at their common boundary forms the mass matrix. The stiffness matrix is constant for a linear-elastic material and is defined as:

\[ k_i = \frac{G_i}{h_i}, \] (12)

where \( G_i \) is the shear modulus and \( h_i \) is the thickness of layer \( i \).
Viscous damping is added, in the form of damping matrix \([C]\), to represent damping at very small strains where many soil models are primarily linear. It is not possible to make the viscous damping frequency independent since in time domain analysis, an arbitrary motion is not decomposed into a finite sum of harmonic components.

### 3. Viscous Damping Matrix Formulation

In time domain analysis, the viscous damping matrix is frequency dependent. The type of damping formulation determines the degree of frequency dependence of the damping. In the original damping formulation proposed by Rayleigh and Lindsay [1945], the \([C]\) matrix is assumed to be proportional to the mass and stiffness matrices:

\[
[C] = a_0[M] + a_1[K].
\]  

(13)

The viscous damping matrix is dependent on stiffness, mass and the natural modes of the soil column. The natural modes and the soil column stiffness are derived from the shear wave velocity profile of the soil column. Early formulations used a simplified form of the damping matrix, which is only stiffness proportional. Hudson [1994] and Hashash and Park [2002] describe the application of the full Rayleigh formulation in site response analysis.

If the damping ratio is constant throughout the soil profile, scalar values of \(a_0\) and \(a_1\) can be computed using two significant natural modes \(m\) and \(n\) using the following equation:

\[
\begin{bmatrix}
\xi_m \\
\xi_n \\
\end{bmatrix}
= \frac{1}{4\pi}
\begin{bmatrix}
\frac{1}{f_m} & f_m \\
\frac{1}{f_n} & f_n \\
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.
\]

(14)

In site response analysis the natural frequency of the selected mode is commonly calculated as [Kramer, 1996]:

\[
f_n = \frac{V_s}{4H} (2n - 1),
\]

(15)

where \(n\) is the mode number and \(f_n\) is the natural frequency of the corresponding mode. It is a common practice to choose frequencies that correspond to the first mode of the soil column and a higher mode that corresponds to the predominant frequency of the input motion [Hudson, 1994; Rathje and Bray, 2001; Borja et al., 2002]. Since the damping ratio has been known to be frequency independent, equal values of modal damping ratios are specified for the two modes.

Equation (14) results in a frequency dependent damping ratio even if the specified damping ratios at the two selected modes are equal, as shown in Fig. 2 [Hashash and Park, 2002]. The frequency dependent damping ratio is referred to as the effective damping ratio in this paper. Equation (13) can be formulated to model a...
multi-layered profile with varying damping ratios, as introduced by Hashash and Park [2002]. The viscous damping matrix that incorporates an updated stiffness matrix in a non-linear analysis is also described in Hashash and Park [2002].

Equation (13) can be extended so that more than two frequencies/modes can be specified, and is referred to as extended Rayleigh formulation. Using the orthogonality conditions of the mass and stiffness matrices, the damping matrix can consist of any combination of mass and stiffness matrices [Clough and Penzien, 1993], as follows:

\[
[C] = [M] \sum_{b=0}^{N-1} a_b [M]^{-1}[K]^b, \quad (16)
\]

where \( N \) is the number of frequencies/modes incorporated.

The coefficient \( a_b \) is a scalar value assuming a constant damping ratio throughout the profile and is defined as follows:

\[
\xi_n = \frac{1}{4\pi f_n} \sum_{b=0}^{N-1} a_b (2\pi f_n)^{2b}. \quad (17)
\]

Equation (16) implies that the damping matrix can be extended to incorporate any number of frequencies/modes. The formulation reduces to the Rayleigh damping formulation, Eq. (13), when \( b = 0 \) to 1. The matrix resulting from Eq. (17) is numerically ill-conditioned since coefficients \( f_n^{-1}, f_n, f_n^3, f_n^5, \ldots \) can differ by orders of magnitude. Having more than four frequencies/modes can result in a singular matrix depending on \( f_n \) and \( a_b \) cannot be calculated.
Addition of a frequency/mode is accompanied by an increase in the number of diagonal bands of the viscous damping matrix, resulting in a significant increase in the numerical cost of the solution. Incorporating an odd number of modes is also problematic since it will result in negative damping at certain frequencies [Clough and Penzien, 1993]. Only four modes are used in this paper. The coefficients \( a_b \) is calculated from Eq. (17) for \( b = 0 \) to 3:

\[
\begin{bmatrix}
\xi_m \\
\xi_n \\
\xi_o \\
\xi_p
\end{bmatrix} = \frac{1}{4\pi} \begin{bmatrix}
\frac{1}{f_m} & f_m^2 & f_m^5 \\
\frac{1}{f_n} & f_n^2 & f_n^5 \\
\frac{1}{f_o} & f_o^2 & f_o^5 \\
\frac{1}{f_p} & f_p^2 & f_p^5
\end{bmatrix} \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix},
\tag{18}
\]

where \( f_m, f_n, f_o, \) and \( f_p \) are selected frequencies and \( \xi_m, \xi_n, \xi_o, \) and \( \xi_p \) are the damping ratios at these frequencies.

The effective damping resulting from Eq. (18) is illustrated in Fig. 2 assuming the damping ratios are equal at the selected frequencies/modes. The damping ratio remains close to the target damping ratio over a wider range of frequencies than the Rayleigh damping formulation and is exact at the four specified frequencies.

For \( b = 0 \) to 3, Eq. (16) can be expanded to model a varying damping ratio in the different soil layers:

\[
[C] = [a_0][M] + a_1[K] + [a_2][[[K][M]^{-1}[K]]] + [a_3]([[K][M]^{-1}[K]][M]^{-1}[K])
\]

\[
= \begin{bmatrix}
a_{01}M_1 & 0 & 0 \\
0 & a_{02}M_2 & 0 \\
0 & 0 & \cdots
\end{bmatrix} + \begin{bmatrix}
a_{11}K_1 & a_{11}K_{12} & 0 \\
a_{12}K_{21} & a_{12}K_{22} & a_{12}K_{23} \\
0 & a_{13}K_{32} & \cdots
\end{bmatrix} + \begin{bmatrix}
a_{21} & 0 & 0 \\
0 & a_{22} & 0 \\
0 & 0 & \cdots
\end{bmatrix}
\]

\[
\times \begin{bmatrix}
\frac{K_{11}^2}{M_1} + \frac{K_{12}K_{21}}{M_2} & \frac{K_{21}K_{11}}{M_1} + \frac{K_{22}K_{21}}{M_2} & \frac{K_{12}K_{23}}{M_2} \\
\frac{K_{21}K_{11}}{M_1} + \frac{K_{22}K_{21}}{M_2} & \frac{K_{21}K_{12}}{M_1} + \frac{K_{22}K_{22}}{M_2} + \frac{K_{22}K_{23}}{M_2} + \frac{K_{23}K_{33}}{M_3} \\
\frac{K_{32}K_{21}}{M_2} & \frac{K_{32}K_{22}}{M_2} + \frac{K_{33}K_{32}}{M_3} & \cdots
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
a_{31} & 0 & 0 \\
0 & a_{32} & 0 \\
0 & 0 & \cdots
\end{bmatrix} \cdot \frac{[K][M]^{-1}[K][M]^{-1}[K]},
\tag{19}
\]

where \( a_{bi} \) represents the coefficient of layer \( i \) and is calculated using Eq. (18).
The extended Rayleigh damping formulation is incorporated in a site response analysis code for the first time in this paper and its effectiveness is evaluated. The mass and stiffness proportional components of the damping $[C]$ matrix, for Rayleigh and extended Rayleigh damping formulations, are multiplied by the relative velocities of the soil layers as given in Eq. (11). The proposed procedures in this paper cannot be used with formulations of dynamic equations of motion whereby total velocity terms are used with mass proportional damping.

4. Selection of Frequencies/Modes for the Rayleigh and Extended Rayleigh Damping Formulations

The frequency dependent nature of the Rayleigh damping formulations implies that the accuracy of the time domain solution depends on the frequencies/modes selected to define the damping function. A series of analyses are presented to examine the influence of the selected frequencies/modes of the Rayleigh damping formulations on the site response analysis and to illustrate how to choose the optimum frequencies/modes. Linear wave propagation analysis is used in the selection process since viscous damping is independent of, but additive to, hysteretic damping resulting from a non-linear material model. The optimum frequencies/modes are selected such that the linear time domain solution (with frequency dependent damping) compares well with the linear frequency domain solution (with frequency independent damping) that represents the correct solution. The selection process using linear analysis is a required step prior to performing a nonlinear site response analysis. In other words, the significant frequencies/modes should be selected through a linear analysis and the chosen frequencies/modes can then be employed in the nonlinear analysis.

Both the frequency and time domain solutions are solved using the site response program DEEPSOIL [Hashash and Park, 2002]. The frequency domain solution scheme implemented in DEEPSOIL is similar to that in SHAKE [Schnabel et al., 1972], without the limitation on the number of layers and material types.

Three soil columns, 100 m, 500 m and 1000 m, representative of the profiles in the Mississippi Embayment, are used (Fig. 3). The corresponding natural frequencies are 1.1 Hz (0.9 sec), 0.35 Hz (2.9 sec), and 0.2 Hz (5.18 sec) for the 100 m, 500 m, and 1000 m columns respectively. In time domain analysis, the thickness of the layer controls the maximum frequency that can be propagated:

$$f_{\text{max}} = \frac{(V_s)_i}{4h_i},$$  

where $f_{\text{max}}$ = maximum frequency that layer $i$ can propagate, $(V_s)_i$ = shear velocity, $h_i$ = thickness of each layer. The layer thickness has been chosen so that $f_{\text{max}}$ is equal to 50 Hz through all layers within the soil column. The 100 m column has 49 layers whereas the 1000 m column has 277 layers.

Two viscous damping profiles are used (Fig. 3). The viscous damping profile (1) has a constant damping ratio of 1.8% and the viscous damping profile (2) has
Fig. 3. Shear wave velocity and viscous damping profiles used in analyses. The profile properties are representative of conditions encountered in the Mississippi Embayment. Central US Bedrock shear wave velocity is 3000 m/sec. Shear wave velocity profile is the generic profile of the Mississippi Embayment developed by Romero and Rix [2001].

a vertical variation of damping ratio, decreasing with depth. The viscous damping profile (2) represents the confining pressure dependent small strain damping from the design soil curves developed by EPRI [1993].

Four input motions, two synthetic (termed S-TS) and two recorded time series (termed R-TS) at rock outcrops, are used (Fig. 4 and Fig. 5). The main characteristics of the motions are summarised in Table 1.

The synthetic ground motions are generated by SMSIM [Boore, 2000] using $M$ (moment magnitude) = 5 − $R$ (epicentral distance) = 20 km (S-TS1) and $M = 8 − R = 32$ km (S-TS2), as shown in Fig. 4. Synthetic ground motions are routinely used in areas such as the Mississippi Embayment due to the absence of strong motion records. SMSIM has been used in the development of the USGS seismic hazard maps in the Central and Eastern United States [Frankel, 1996]. For S-TS1, ($M = 5 − R = 20$ km), the ground motion content is concentrated in a limited frequency range of 2−15 Hz, whereas the content is evenly distributed over the frequency range of 1−50 Hz in S-TS2 ($M = 8, R = 32$ km motion).

R-TS1 is the recorded motion during the 1985 Nahanni earthquake ($M = 6.9, R = 16$ km), in the Northwest Territories of Canada, as shown in Fig. 5. The predominant frequency of the motion is about 16 Hz and has limited content below 1 Hz. R-TS2 is an E-W recording at Yerba Buena Island during the 1989 Loma
Fig. 4. Synthetic motions generated using SMSIM: (a) Time history and (b) Fourier amplitude of Boore [2000]. S-TS1: \( M = 5, R = 20 \) km, S-TS2: \( M = 8, R = 32 \) km.

Fig. 5. Recorded ground motions from rock sites: (a) Time history and (b) Fourier amplitude. R-TS1: recording at station 6099 during the Nahanni Earthquake (1985), and R-TS2: E-W recording at Yerba Buena Island during Loma Prieta Earthquake (1989).

Prieta earthquake (\( M = 6.9, R = 80.6 \) km), in northern California (Fig. 5). Contrary to other motions, the highest concentration of the ground motion content is at low frequencies, from 0.1 to 1 Hz.

Linear wave propagation analyses are performed using selected combinations of the four motions and the three soil columns. The purpose of the analyses is to (1) investigate how to choose the significant frequencies/modes in RF and ERF,
Table 1. Summary of ground motions used in response analyses.

<table>
<thead>
<tr>
<th>Ground Motion Number</th>
<th>Location and Event</th>
<th>$a_{\text{max}} (g)$</th>
<th>Dominant Frequency (range)</th>
<th>Linear Frequency and Time Domain Analyses</th>
<th>Equivalent Linear/Non-Linear Analyses</th>
<th>Soil Column Thickness (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Synthetic ground motion (S-TS) generated using SMSIM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S-TS 1</td>
<td>$M = 5, R = 20$ km</td>
<td>0.059</td>
<td>4 (2–15)</td>
<td>×</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>S-TS 2</td>
<td>$M = 7, R = 32$ km</td>
<td>0.653</td>
<td>1* (1–50)</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Recorded ground motion (R-TS) from rock sites</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R-TS 1</td>
<td>Station 6099, Nahanni</td>
<td>0.148</td>
<td>16 (10–20)</td>
<td>×</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Earthquake (1985)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R-TS 2</td>
<td>Yerba Buena Island,</td>
<td>0.067</td>
<td>0.7 (0.1–1)</td>
<td></td>
<td>×</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Loma Prieta Earthquake (1989)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TS: Time series also referred to as time history.

*It is not meaningful to define a predominant frequency as the motion content is evenly distributed over a wide frequency range.
compare the accuracy of RF and ERF, and (3) demonstrate the effect of the depth of the soil profile on the accuracy of the time domain solution.

4.1. *S-TS1, 100 m soil column*

Analyses are performed using $M = 5 - R = 20$ km synthetic motion, 100 m, 500 m, and 1000 m soil columns, and the viscous damping profile (1). Three analyses are performed for each profile, two using RF and one using ERF, and the results are compared to the frequency domain solution. The first RF analysis follows the conventional guideline for selecting frequencies. The 1st frequency corresponds to the first mode/natural frequency of the soil column, calculated using Eq. (15), and the 2nd frequency corresponds to the predominant frequency of the input motion [Hudson, 1994; Rathje and Bray, 2001; Borja et al., 2002] or a higher mode. Such solution will be termed CRF in this study. In the second RF analysis and the ERF analysis, significant frequencies are chosen independent of soil column modes with the aim of obtaining the best match with the frequency domain solution.

The results for the 100 m profile are shown in Fig. 6. Figure 6(a) shows the Fourier spectrum ratio of the computed surface motion to the input motion. Note that the Fourier spectrum ratio for the frequency domain solution is equivalent to the transfer function shown in Eq. (6). Figure 6(b) plots the effective damping ratio for the CRF, RF and ERF analyses. Figure 6(c) plots the 5% damped elastic response spectrum of the computed surface motion.

The CRF effective damping ratio underestimates the target damping ratio between 1.1 and 4 Hz (corresponding to 1st and 2nd modes of the soil column) and overestimates the target damping ratio at other frequencies, as shown in Fig. 6(b). The Fourier spectrum ratio, Fig. 6(a), and response spectrum, Fig. 6(c), show that CRF underestimates the surface motion at frequencies greater than 4 Hz. CRF analysis results in significant filtering of ground motion within a frequency range relevant to engineering applications.

The performance of the RF analysis can be improved by a different selection of the two significant frequencies. The frequencies should be selected in part to cover the range of significant frequencies in the input motion. For S-TS1, the optimal range chosen is 2–10 Hz through a trial and error process to obtain the best possible match with the Fourier and response spectra of the frequency domain solution. Although RF does not capture the full range of frequency amplification, as shown in Fig. 6(a), the response spectrum agrees well with the frequency domain solution, Fig. 6(c).

High frequency components of the surface motion can be preserved with the ERF solution. The first two frequencies chosen are identical to the ones used in RF. For 3rd and 4th frequencies, 35 and 45 Hz are selected respectively. The ERF solution captures the high frequency components much better than RF, Fig. 6(a), since the effective damping ratio does not increase linearly at frequencies higher than the selected 2nd frequency (10 Hz in this analysis). However, the 5% damped response spectrum is very similar for both RF and ERF.
4.2. **S-TS1, 500 m soil column**

An increase in the depth of the soil column is accompanied by a decrease in the natural frequency of the soil column and an increase in the number of resonant modes as illustrated by the frequency domain solution in Fig. 7(a). Using CRF, the frequencies selected are 0.35 and 4 Hz. At 10 Hz, the effective damping ratio for the 100 m profile is 3.7% whereas it is 4.2% for the 500 m profile. Therefore, the CRF
for the 500 m column introduces higher damping compared to the 100 m column at frequencies higher than 4 Hz.

Using 2 and 10 Hz in the RF, the match with the frequency domain solution is significantly improved at higher frequencies, as shown in Fig. 7(a). The lower estimate at low frequencies (the first two modes) is not as important due to the low input motion content up to 2 Hz. A comparison of the 5% damped elastic
response spectra shows the good match with the frequency domain solution. The ERF solution, using frequencies of 2, 10, 35, 45 Hz, results in a slightly better match with the frequency domain solution.

4.3. **S-TS1, 1000 m soil column**

The 1000 m column is most sensitive to overestimation or underestimation of the effective damping ratio due to a significant increase in number of resonant modes.

![Graphs showing Fourier spectrum ratio, effective damping ratio, and spectral acceleration](image)

**Fig. 8.** Linear frequency and time domain analyses for 1000 m soil column, damping profile (1), input motion S-TS1, (a) Fourier spectrum ratio, (b) effective damping ratio, and (c) 5% damped elastic response spectra.
A decrease in the natural frequency of the soil column (0.2 Hz) is accompanied by further widening of the effective damping ratio curve for CRF, as shown in Fig. 8(b). The CRF solution highly underestimates the high frequency motion. The RF solution using frequencies of 2 and 7 Hz shows a better match with the frequency domain solution at higher frequencies [Fig. 8(a)], however the mismatch at lower frequencies increases. Overall the solution improves significantly when the elastic response spectra are compared in Fig. 8(c). The 2nd frequency has been reduced

![Graphs showing Fourier spectrum ratio, effective damping ratio, and spectral acceleration.](image)

Fig. 9. Linear frequency and time domain analyses for 100 m soil column, damping profile (1), input motion R-TS1, (a) Fourier spectrum ratio, (b) effective damping ratio, and (c) 5% damped elastic response spectra.
from 10 to 7 Hz since the frequency range where the motion is amplified (Fourier spectrum ratio $> 1$) has decreased [Fig. 8(a)]. The use of ERF results in some but limited improvement of the results.

### 4.4. R-TS1 100 m and 1000 m columns

The predominant frequency of the R-TS1 input motion is 16 Hz. For the 100 m profile, CRF (1.1 and 16 Hz) slightly overestimates the response but overall provides...
a reasonable match with the frequency domain solution, as shown in Figs. 9(a) and (c). The match is improved when using RF with 3 and 16 Hz. ERF (3, 16, 35, 45 Hz) provides a better match for Fourier spectrum ratio at high frequencies, but again the difference is small when comparing the 5% damped elastic response spectra.

For the 1000 m profile, Fig. 10, CRF significantly overestimates the response due to underestimation of the effective damping ratio (0.2 to 16 Hz). The RF solution using 1.5 and 6.5 Hz and ERF using 1.5, 6.5, 35 and 45 Hz compare well with the frequency domain solution when the elastic response spectra are compared.

4.5. S-TS2 and R-TS2 1000 m column

Figure 11 shows a comparison of the surface elastic response spectra for the 1000 m column using input motions R-TS2 and S-TS2. Both analyses show that the use of the CRF results in a significant underestimation of the surface response. The RF and ERF solutions whereby the significant frequencies are chosen through an iterative procedure to match the frequency domain solution provide a significantly improved response.

![Figure 11](image-url)

Fig. 11. The 5% damped elastic response spectra from linear frequency and time domain analyses for 1000 m soil column, damping profile (1), (a) input motion R-TS2, and (b) input motion S-TS2.
4.6. Effect of variable damping profile

Figure 12 presents analysis results for the 1000 m column using S-TS2 and damping profile (2) to investigate the effect of varying the viscous damping ratio throughout the profile and the validity of Eq. (19). Profile (2) has a much lower damping, in comparison to viscous damping profile (1), and it accounts for the effect of confining pressure on small strain damping.

The CRF solution significantly underestimates the surface response. RF (1.0 and 6.4 Hz) and ERF (1.0, 6.4, 35, 45 Hz) provide a better match with the frequency domain solution especially when the elastic response spectra are compared. Nevertheless, the match is not exact because the RF and ERF solution still result in frequency dependent damping. The linear analyses presented show that:

- CRF may underestimate or overestimate the ground motion response at high frequencies. The accuracy of the solution deteriorates with an increase in the depth of the soil column.
- ERF provides the best match with the frequency domain solution, but is computationally expensive (an ERF analysis can be 1.3 to 10 times slower than a RF analysis, depending on the profile and input motion).
The 5% damped surface elastic response spectra computed with the RF and ERF using the proposed iterative procedure for selecting the significant frequencies compare well with the frequency domain solution.

For all ERF analyses performed in this study, constant 3rd and 4th frequencies of 35 and 45 Hz respectively are appropriate.

5. Nonlinear Wave Propagation Analysis and Comparison with Equivalent Linear Analysis

The development of RF and ERF and corresponding significant frequencies/modes selection procedure is introduced to enhance the accuracy of the time domain

![Graph showing shear modulus and damping ratio vs. shear strain for different confining pressures.](image)

<table>
<thead>
<tr>
<th>Confining pressure</th>
<th>Laird &amp; Stoke (measured)</th>
<th>Non-linear pressure dependent model</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.6 kPa</td>
<td>□</td>
<td>1=27.6 kPa 2=55.2 kPa 3=110 kPa</td>
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<tr>
<td>55.2 kPa</td>
<td>×</td>
<td>4=221 kPa 5=442 kPa 6=883 kPa</td>
</tr>
<tr>
<td>883 kPa + 442 kPa</td>
<td>▲</td>
<td>7=1776 kPa 8=10 MPa</td>
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<tr>
<td>1776 kPa</td>
<td>△</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameters for the modified hyperbolic equation</th>
<th>Value</th>
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<tr>
<td>$\beta$</td>
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<tr>
<td>$s$</td>
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<tr>
<td>$\sigma_{ref}$</td>
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<tr>
<td>$a$</td>
<td>0.163</td>
</tr>
<tr>
<td>$b$</td>
<td>0.63</td>
</tr>
</tbody>
</table>

Fig. 13. Influence of confining pressure on (a) shear modulus degradation and (b) damping ratio curves in DEEPSOIL nonlinear model used for site response analyses in the Mississippi Embayment. Data from Laird and Stokoe [1993] shown for comparison.
non-linear site response solution. Nonlinear site response analyses are presented using S-TS2 and the 1000 m soil column. The RF and ERF frequencies used are those selected in the linear analysis shown in Fig. 11(b) and Fig. 12.

5.1. Non-linear soil model

The pressure dependent non-linear soil model described by Hashash and Park [2001] is used. The model is an extension of the modified hyperbolic model developed by [Matasovic, 1993], and accounts for the influence of confining pressure on soil dynamic properties. The model is described by the following equation:

\[ \tau = \frac{G_{mo} \gamma}{1 + \beta \left( \frac{\sigma}{\sigma_{ref}} \right)} \tau , \quad \text{where } \gamma_r = a \left( \frac{\sigma}{\sigma_{ref}} \right)^b, \]  

where \( \tau \) = shear stress, \( \gamma \) = shear strain, \( G_{mo} \) = initial shear modulus, \( b \) and \( s \) = curve fitting parameters that adjust the shape of the backbone curve, \( \gamma_r \) is

\[ \tau = \frac{G_{mo} \gamma}{1 + \beta \left( \frac{\sigma}{\sigma_{ref}} \right)} \tau , \quad \text{where } \gamma_r = a \left( \frac{\sigma}{\sigma_{ref}} \right)^b, \]  

Fig. 14. Total soil damping equal to the sum of hysteretic damping from modified hyperbolic model of Fig. 13, and (a) viscous damping profile (1), or (b) viscous damping profile (2).
the reference shear strain [Hardin and Drnevich, 1972], \( a \) and \( b \) are curve fitting parameters to account for confining pressure dependent soil behavior, and \( \sigma_{\text{ref}} \) is a reference confining pressure. Model properties are chosen such that the resulting modulus degradation and damping curves approximate the laboratory test data of Laird and Stokoe [1993], as shown in Fig. 13.

5.2. Small strain damping

Figure 13(b) shows that the hysteretic damping from the hyperbolic model is negligible at small strains and underestimates damping values obtained from laboratory measurements. The damping profiles (1) and (2) are used to represent the

![Diagram](attachment:image.png)

Fig. 15. The 5% damped response spectra of equivalent linear and nonlinear analyses for 1000 m soil column, input motion S-TS2 (a) using damping profile (1) and (b) damping profile (2).
small strain damping and are added as viscous damping quantities in the following analyses. Figure 14 shows the combined viscous and hysteretic damping curves represented in the site response analysis that result from the addition of viscous damping.

Figure 15 shows the response spectra of the nonlinear analyses performed. The CRF solution results in pronounced underestimation of high frequency components up to 0.7 sec. RF and ERF solutions are almost identical, except at short periods for viscous damping profile (2), where ERF shows slightly higher response than RF. ERF also shows higher response at periods greater than 4 sec for both viscous damping profiles.

The nonlinear response analysis results are compared to the equivalent linear solution. Identical modulus degradation and damping are used. Equivalent linear analysis is often used to approximate the non-linear soil response. Equivalent linear analysis uses constant linear shear modulus and damping throughout the duration of the motion selected at a representative level of strain. For both profiles, the equivalent linear solution underestimates the response at periods less than 0.3 sec. The equivalent linear analysis filters these higher frequency components. Nonlinear analysis is able to preserve such high frequency components. The equivalent linear analysis also underestimates the response at longer periods. If equivalent linear analysis is compared to the nonlinear analysis with CRF, it results in the misleading conclusion that the equivalent linear solution is comparable or provides larger estimates than the nonlinear solution at most period ranges.

The analyses demonstrate the importance of accurately selecting the viscous damping formulation in a nonlinear site response analysis.

6. Guidelines for Performing Non-Linear Site Response Analyses with Viscous Damping

A guideline is proposed for performing a nonlinear analysis with viscous damping:

(1) Layer thickness: Use Eq. (20) to determine the maximum layer thickness and thus avoid filtering of relevant frequencies.

(2) Viscous damping formulation: Select the appropriate frequencies for RF or ERF formulation. The selection is performed through an iterative procedure and by comparing the results of a linear time domain analysis with the linear frequency domain solution. RF formulation is adequate if the elastic response spectrum is needed. ERF may be necessary if the Fourier spectral content is desired. The ERF is computationally expensive compared to RF. For many engineering applications, RF will provide acceptable results. Care should be exercised when selecting frequencies to avoid negative damping in the resulting frequency dependent damping.

(3) Rayleigh damping formulation: The conventional guideline of using the first and a higher mode of the soil column or the predominant period of the input
motion will not always result in a good match with the linear frequency domain solution especially for deep soil columns. The two significant frequencies can be chosen in part to cover the range of frequencies where there is significant input motion content. A new set of frequencies will have to be selected if a different input motion is selected.

(4) Extended Rayleigh damping formulation: Four significant frequencies are used. The first two are usually identical to the two frequencies chosen for RF. The spacing between the 3rd and 4th frequencies should be approximately 10 Hz. If a higher spacing is used, it can result in a negative damping ratio. In this paper acceptable results for all ground motions and soil profiles are obtained when the 3rd and 4th frequencies are equal to 35 Hz and 45 Hz respectively. ERF improves the Fourier spectrum match by preserving frequencies higher than the 2nd frequency. However, such an improvement over RF has negligible influence when calculating the 5% damped elastic response spectrum.

(5) Non-linear analysis: Once the appropriate frequencies for the viscous damping formulation are chosen based on the linear analysis, a full non-linear analysis can be performed.

This selection process will have to be repeated if a different input motion or soil column properties are used.

7. Conclusions

One-dimensional non-linear site response analysis is performed using two types of Rayleigh viscous damping formulations, (a) Rayleigh damping formulation (RF) and (b) extended Rayleigh damping formulation (ERF). Both RF and ERF formulations result in frequency dependent damping, even though the actual damping of soils is considered frequency independent.

The choice of significant frequencies/modes for RF and ERF has a pronounced influence on the computed surface response in a nonlinear analysis. Prior to running the nonlinear analysis, linear analyses should be performed to obtain optimum frequencies/modes. Linear time domain solutions are compared to the frequency domain solution. New guidelines taking into account the frequency content characteristics of the input motion and the soil column properties are proposed. By choosing appropriate frequencies/modes, both RF and ERF agree well with the frequency domain solution. ERF provides a better match when comparing the Fourier spectra, but its effect on the 5% damped response spectrum is limited. Nonlinear analyses are performed using both RF and ERF. RF using conventional procedures underestimates the surface response. Comparisons of the nonlinear and equivalent linear solutions demonstrate that the equivalent linear solution filters out high frequency components of the input motion and underestimates surface ground motion at long periods.
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